Axiomatization of Differential Cohomology

Algebra Seminar

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Motivation

$$H^k(M,\mathbb{Z}) \longrightarrow H^k(M,\mathbb{R}) \cong H^k_{dR}(M)$$

Motivation

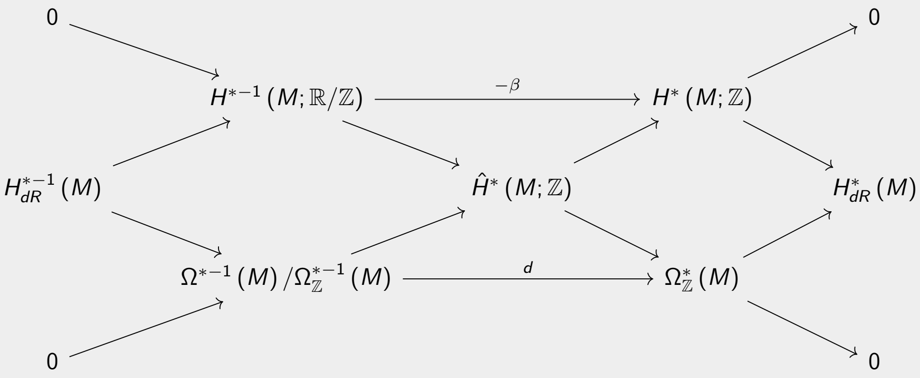
$\forall c \in Z_k(M;\mathbb{R})$ and $\omega \in \Omega^k_{cl}$,

$$\int_c \omega \in \mathbb{Z}$$

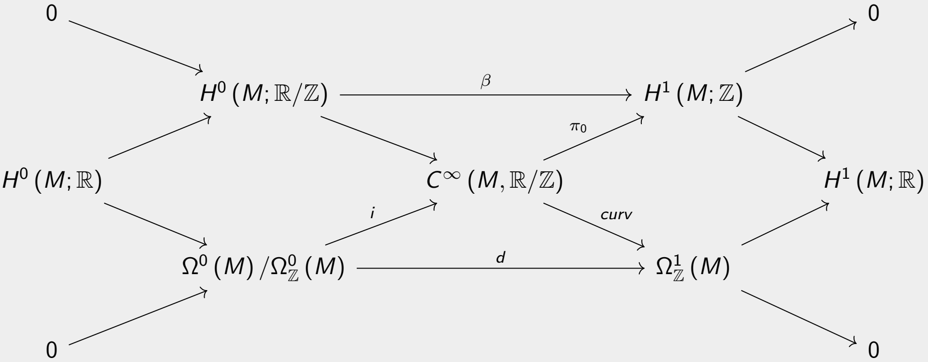
if and only if $[\omega] \in H^k(M,\mathbb{R})$ lies in the image of the map

$$H^k(M,\mathbb{Z}) \longrightarrow H^k(M,\mathbb{R}) \cong H^k_{dR}(M)$$

The Differential Cohomology Diagram

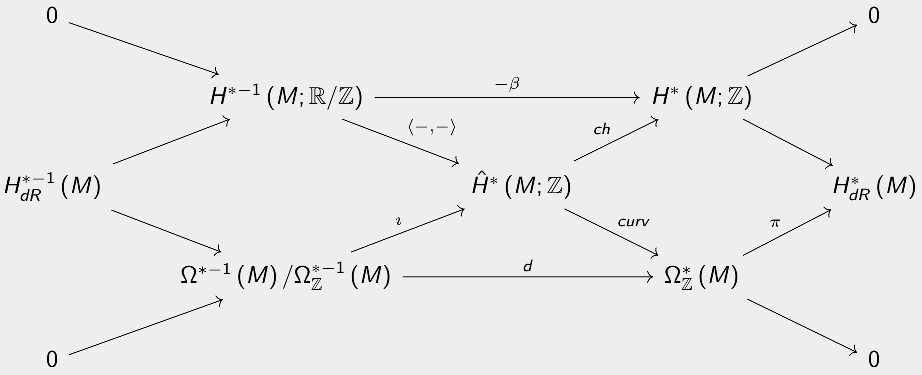


Case for $k = 1$



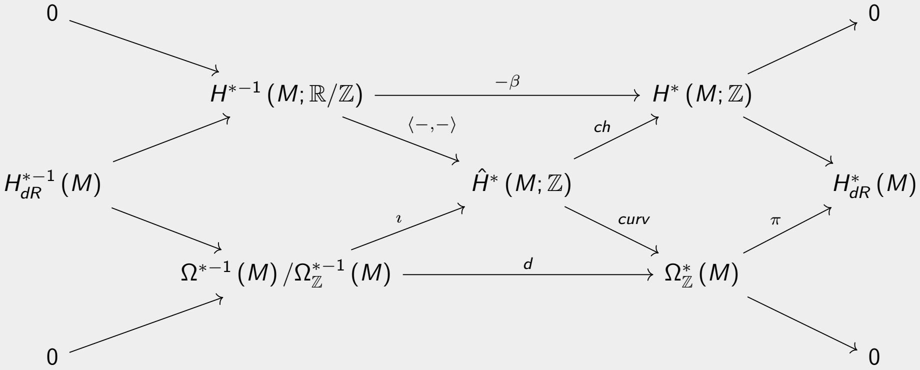
0
0
$H^0(M; \mathbb{R}/\mathbb{Z})$
β
$H^1(M; \mathbb{Z})$
π_0
$H^0(M; \mathbb{R})$
$C^\infty(M, \mathbb{R}/\mathbb{Z})$
$H^1(M; \mathbb{R})$
i
$curv$
$\Omega^0(M)/\Omega^0_{\mathbb{Z}}(M)$
d
$\Omega^1_{\mathbb{Z}}(M)$
0
0

In general degree..



$$\hat{H}^k(M;\mathbb{Z}) = \left\{ \begin{array}{c} \chi \in Hom(Z_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \mid \exists \omega \in \Omega^k \text{ such that} \\ \chi(\partial -) = \int_{\_} \omega \text{ mod } \mathbb{Z} \end{array} \right\}$$

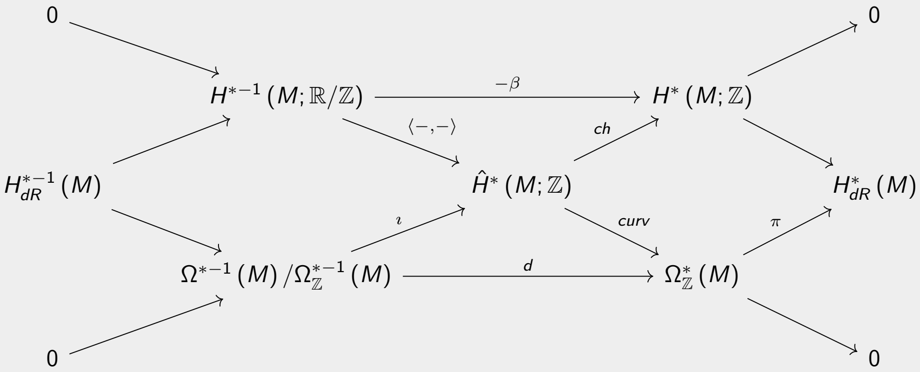
In general degree..

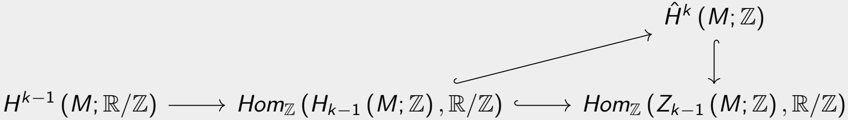


$$\hat{H}^{k}(M;\mathbb{Z}) = \left\{ \begin{array}{c} \chi \in Hom\left(Z_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z}\right) \mid \exists \omega \in \Omega^{k} \text{ such that} \\ \chi(\partial -) = \int_{-} \omega \text{ mod } \mathbb{Z} \end{array} \right\}$$

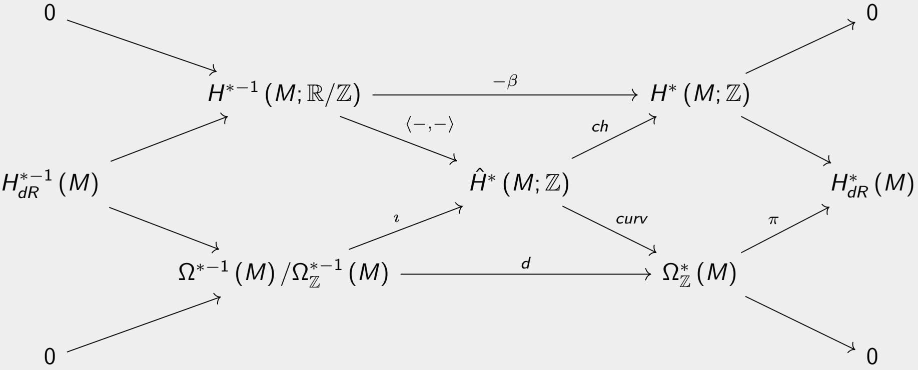
$$ch(\chi) = [I(\overline{\chi})] \text{ where } I(\overline{\chi})(c) = \int_{c} curv(\chi) - \chi(\partial c)$$

In general degree..





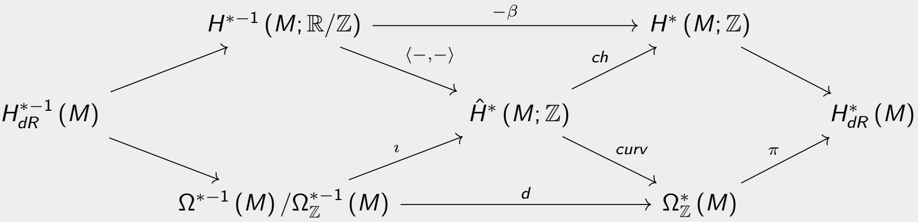
In general degree..



Define $\imath : \Omega^{k-1}(M) \longrightarrow Hom_{\mathbb{Z}}(Z_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ by

$$\imath(\omega)(z) := \exp\left(2\pi i \int_z \omega\right)$$

The Differential Cohomology Diagram



Let $\mathcal{M}an$ be the category of smooth manifolds and $\mathcal{G}r\mathcal{A}b$, the category of graded, abelian groups. There is a unique functor[SS08] $\hat{H}(-;\mathbb{Z}) : \mathcal{M}an^{op} \longrightarrow \mathcal{G}r\mathcal{A}b$ equipped with natural transformations

- $\langle -,- \rangle : \hat{H}^{*-1}(-;\mathbb{R}/\mathbb{Z}) \longrightarrow \hat{H}^*(-;\mathbb{Z})$,
- $\imath : \Omega^{*-1}(-)/\Omega^{*-1}_{\mathbb{Z}}(-) \longrightarrow \hat{H}^*(-;\mathbb{Z})$,
- $ch : \hat{H}^*(-;\mathbb{Z}) \longrightarrow \hat{H}^*(-;\mathbb{Z})$, and
- $curv : \hat{H}^*(-;\mathbb{Z}) \longrightarrow \Omega^{*-1}_{\mathbb{Z}}(-)$

satisfying the differential cohomology hexagon

References

- Shiing-Shen Chern and James Simons, *Characteristic forms and geometric invariants*, Annals of Mathematics **99** (1974), no. 1, 48–69.
- Jeff Cheeger and James Simons, *Differential characters and geometric invariants*, Geometry and topology, 1985, pp. 50–80.
- Peter J Haine, *Differential cohomology seminar overview*, 2019.
- James Simons and Dennis Sullivan, *Axiomatic characterization of ordinary differential cohomology*, Journal of Topology **1** (2008), no. 1, 45–56.

Some notes at `https://www.academia.edu/59191057/Notes_on_ordinary_differential_cohomology`